Dynamical Systems

Exam 23 January 2023

university of groningen

The exam consists of 4 questions. You have 120 minutes to do the exam. You can achieve 50 points in total which includes a bonus of 5 points.

1. **[3+3+3=9 Points]**

   Each of the following time-continuous one-dimensional systems depends on a parameter $a \in \mathbb{R}$. Describe the bifurcations involved, sketch the corresponding bifurcation diagrams including representative one-dimensional phase portraits, and classify the bifurcations.

   (a) $x' = x^2 + ax$

   (b) $x' = ax - x^3$

   (c) $x' = x^3 - x - a$

2. **[8 Points]**

   Consider the planar systems

$$X' = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix} X$$

   with parameters $a, b \in \mathbb{R}$. Sketch the regions in the $a - b$ plane where this system has different types of canonical forms. In each region give the canonical form and sketch the phase portrait of the system in canonical form.

3. **[1+2+4+4+2=13 Points]**

   Consider the planar system

$$\begin{aligned} x' &= y, \\ y' &= -\nu y - 4x^3 + 4x, \end{aligned}$$

   where $\nu \geq 0$ is a parameter.

   (a) Show that the system has the three equilibrium points $(x_-, y_-) = (-1, 0)$, $(x_0, y_0) = (0, 0)$ and $(x_+, y_+) = (1, 0)$.

   (b) Show from the linearization at $(x_0, y_0) = (0, 0)$ that this equilibrium is a saddle.

   (c) Show that for $\nu = 0$, the system is Hamiltonian with Hamilton function

$$H(x, y) = \frac{1}{2}y^2 + x^4 - 2x^2 + 1$$

   and sketch the phase portrait in the $x - y$ plane.

   (d) Show that for $\nu \geq 0$ and each $0 < h < 1$, H is a Lyapunov function in the region $D_h = \{(x, y) \in \mathbb{R}^2 \,|\, H(x, y) \leq h,\ x < 0\}$ and use the Lasalle Invariance Principle to show that for $\nu > 0$, the equilibrium at $(x_-, y_-) = (-1, 0)$ is asymptotically stable with D_h belonging to the basin of attraction.

(e) Sketch the phase portrait for $\nu > 0$ by paying attention to the stable and unstable curves of the saddle at $(x_0, y_0) = (0, 0)$. What can you say about the basin of attraction of $(x_-, y_-) = (-1, 0)$.

4. **[9+6=15 Points]**

(a) Show by direct proof (i.e. without using a conjugacy) that the discrete-time system $x_{n+1} = t(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, defined by the tent map

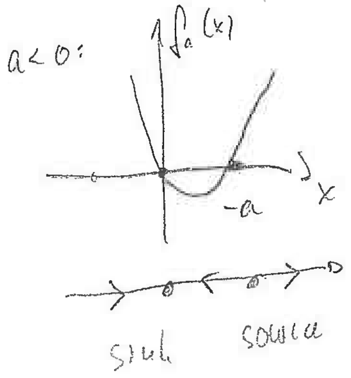
$$t : [0,1] \to [0,1], \quad x \mapsto \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

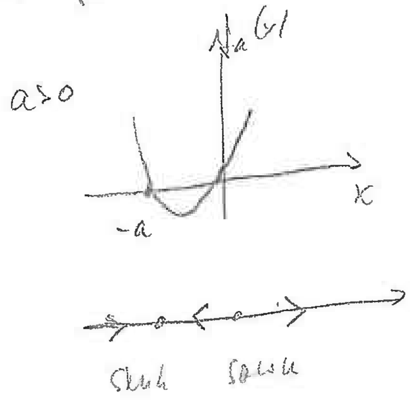
satisfies all three conditions of Devaney's definition of chaos.

(b) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be compact intervals and suppose the two discrete-time systems $x_{n+1} = f(x_n)$ and $y_{n+1} = g(y_n)$ defined by maps $f : I \to I$ and $g : J \to J$ are topologically conjugate. Show that if the discrete-time system $x_{n+1} = f(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, is topologically transitive, then the discrete-time system $y_{n+1} = g(y_n)$, $n \in \mathbb{Z}_{\geq 0}$, is also topologically transitive.

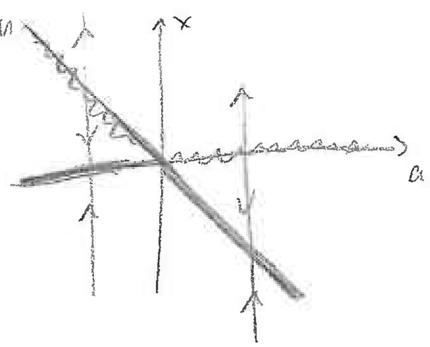
1. $x' = f_a(x)$.

(a) $f_a(x) = x^2 + ax$ equilibria: $f_a(x) = 0 \iff x = 0 \cup x = -a$





bifurcation diagram



= : stable

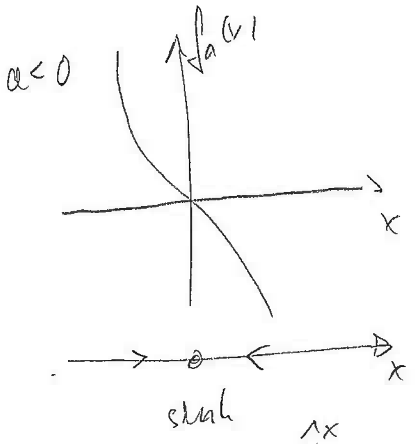
~ : sources

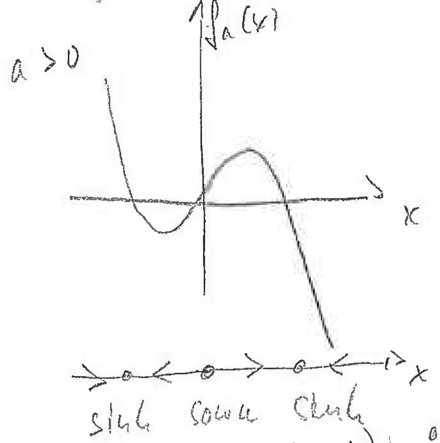
transcritical bifurcation.
two equilibria collide and exchange their stability.

(b) $f_a(x) = ax - x^3$ equilibria: $f_a(x) = 0 \iff x = 0 \cup x^2 = a$

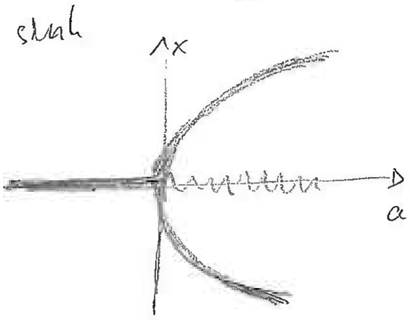
$\iff x = 0 \cup x = \pm\sqrt{a}$

the latter are real for $a \geq 0$





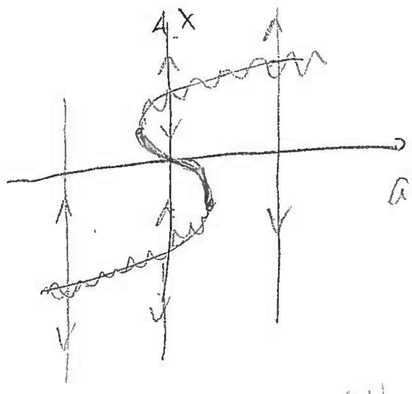
bifurcation diagram

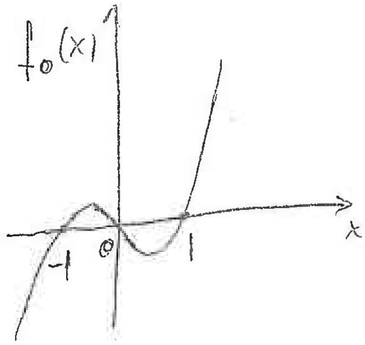


(supercritical) pitchfork bifurcation
stable equilibrium becomes unstable and gives birth to two stable equilibria

(c) $f_a(x) = x^3 - x - a$ equilibria: $a = x^3 - x$

bifurcation diagram





two saddle node bifurcations
where two equilibria of opposite stability
collide and get extinct

2. let $A = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}$.

characteristic polynomial: $p(\lambda) = (a-\lambda)^2 - b$

eigenvalues: $(a-\lambda)^2 = b$

$\Leftrightarrow \quad a - \lambda_\pm = \pm\sqrt{b}$

$\Leftrightarrow \quad \lambda_\pm = a \pm \sqrt{b}$

$b < 0$: eigenvalues complex with $\operatorname{Re}\lambda_\pm = a$

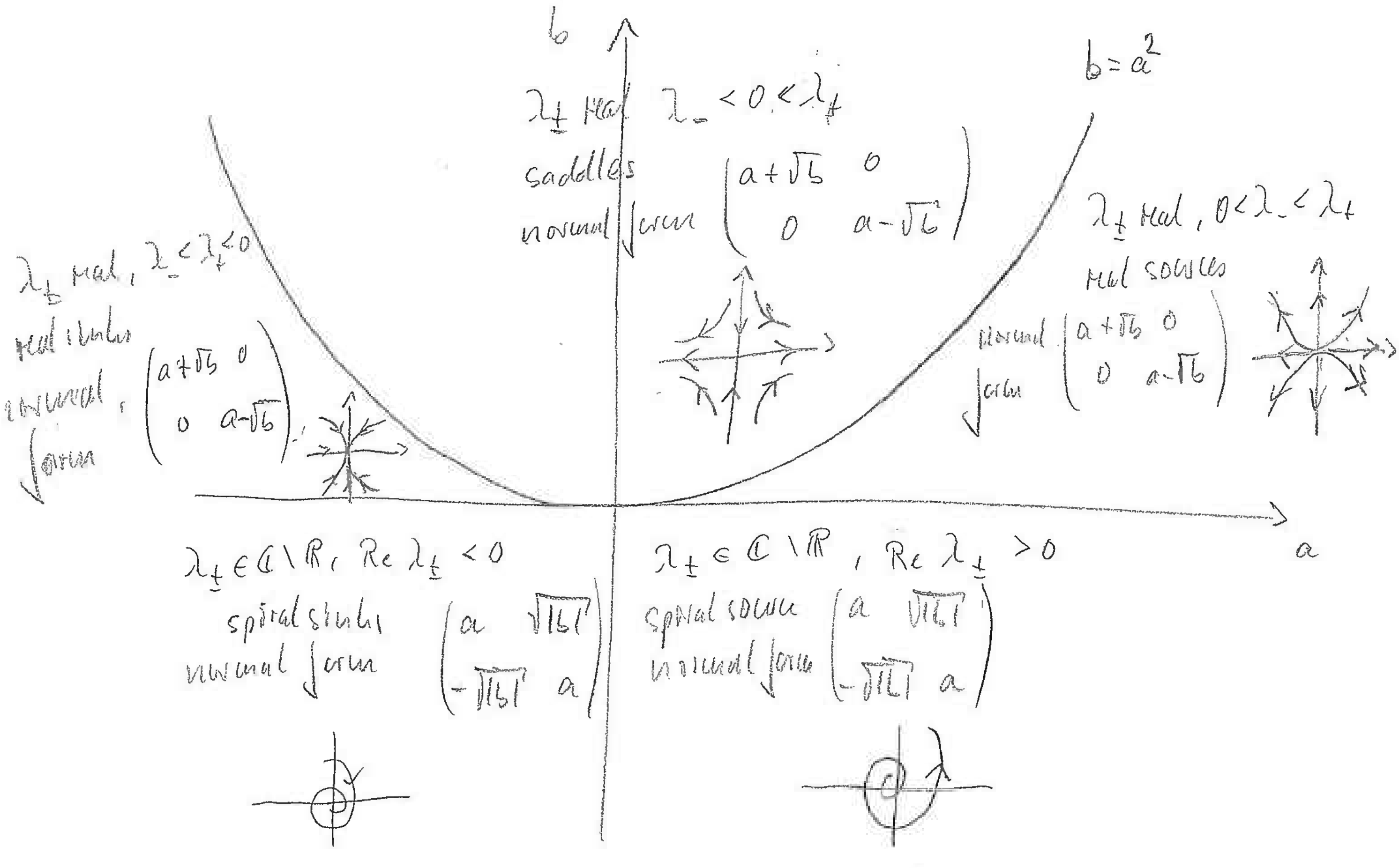
$a < 0$: spiral sink

$a > 0$: spiral source

normal form $\begin{pmatrix} a & \sqrt{|b|} \\ -\sqrt{|b|} & a \end{pmatrix}$

$b > 0$: eigenvalues real $\quad \lambda_+ > 0 \Leftrightarrow a > -\sqrt{b}$

$\lambda_- < 0 \Leftrightarrow a < \sqrt{b}$



3. (a) equilibria: $x' = 0 \Leftrightarrow y = 0$

Then: $y' = -4x^3 + 4x$ which vanishes

for $x = 0$ or $x = \pm 1$

Three equilibria $(x_-, y_-) = (-1, 0)$
$(x_+, y_+) = (1, 0)$
$(x_0, y_0) = (0, 0)$

(b) Linearization at (x_0, y_0) gives matrix

$\begin{pmatrix} 0 & 1 \\ 4 & -\nu \end{pmatrix}$ which has eigenvalues

$$-\lambda(-\nu - \lambda) - 4 = 0$$

$$\Leftrightarrow \lambda_\pm = -\frac{\nu}{2} \pm \sqrt{4 + \left(\frac{\nu}{2}\right)^2}$$

it always holds that $\lambda_- < 0 < \lambda_+$
which gives a saddle

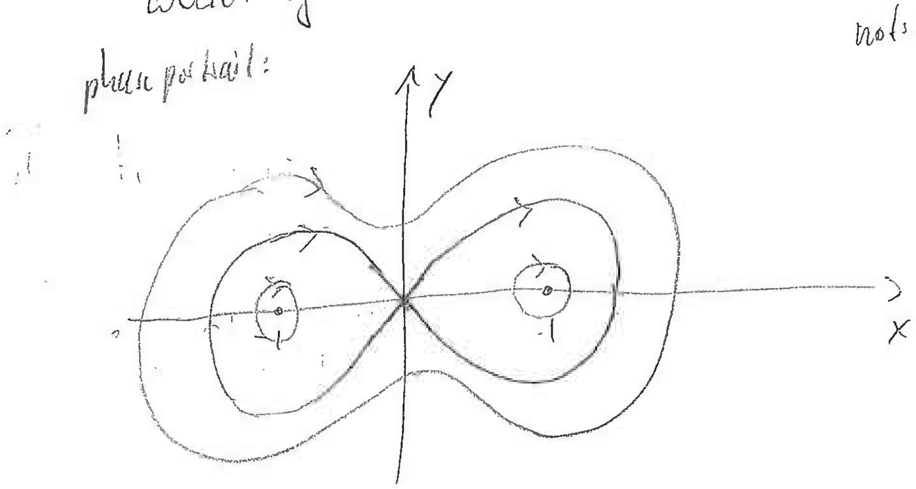
(c) to be shown

$$x' = \frac{\partial H}{\partial y} = y$$

$$y' = -\frac{\partial H}{\partial x} = -4x^3 + 4x$$ which agrees with the given vector field for $\nu = 0$

phase portrait:

note solution curves are level sets of H.



(d) $H(-1,0) = 0$

$H(0,0) = 1$

$$\dot{H} = \frac{\partial H}{\partial x} x' + \frac{\partial H}{\partial y} y' = -\nu y^2 \leq 0$$

$\Rightarrow$ H Lyapunov function on D_h.

D_h is compact and positively invariant

as D_h is enclosed by a level set of H and

H cannot increase along solution curves in the forward

time direction.

D_h contains no entire solutions on which H is constant

besides the equilibrium solution $(-1,0)$:

$\dot{H} \equiv 0 \Leftrightarrow y(t) \equiv 0$ f.a. t

Then $x'(t) = y(t) \equiv 0$ f.a. t

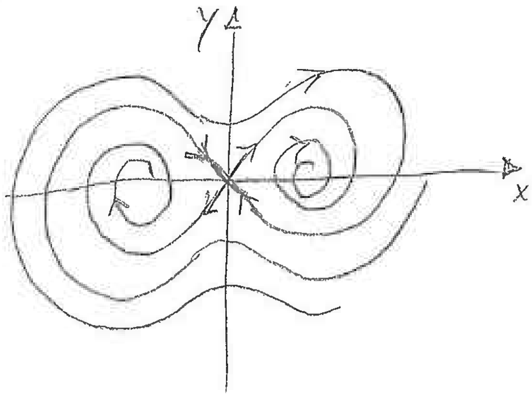
$\Rightarrow x(t) = $ const. f.a. t

But the only solution $(x(t), y(t)) \equiv$ (const., 0)

is the equilibrium solution $(x(t), y(t)) \equiv (-1,0)$.

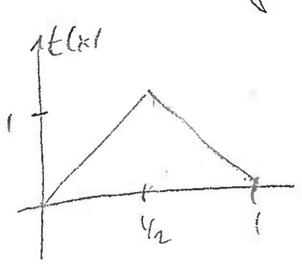
By Lasalle Invariance Principle, (x_-, y_-) is asymptotically stable and D_h is part of the basin of attraction

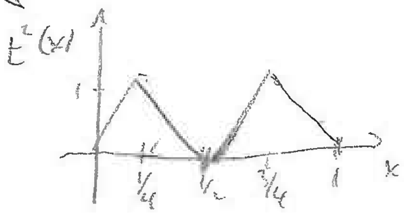
(e)



The basin of attraction is bigger than D_h.

4.(a) Consider the graph of t and its iterates:





t^n maps the 2^n intervals

$$I_k^n := \left[(k-1)\left(\tfrac{1}{2}\right)^n, k\left(\tfrac{1}{2}\right)^n\right], \quad k = 1, \dots, 2^n$$

surjectively to the interval $[0,1]$

We have $[0,1] = \bigcup_{k=1}^{2^n} I_k^n$ and length of I_k^n equal to $\left(\frac{1}{2}\right)^n$

1. Each interval I_k^n contains periodic point
   $\Rightarrow$ periodic points are dense.

2. Let $U, V \subset [0,1]$ open.
   $\Rightarrow \exists\, n, k$ such that $I_k^n \subset U$.
   $$t^n(I_k^n) = [0,1].$$
   recall
   $\Rightarrow \exists x \in I_k^n \subset U$ s.t. $t^n(x) \in V$
   $\Rightarrow$ t transitive

3. Choose $\beta = \frac{1}{2}$. Let $x \in [0,1]$ and U be open neighb. of x.
   to be shown: $\exists\, y \in U$ and $n \in \mathbb{Z}_{\geq 0}$ such that
   $$|t^n(x) - t^n(y)| > \beta$$
   to this end note that $\exists\, n, k \in \mathbb{Z}_{>0}$ s.t. $x \in I_k^n$ and $I_k^n \subset U$. As $t^n(I_k^n) = [0,1]$, there is $y \in I_k^n$ s.t.
   $$|t^n(x) - t^n(y)| > \tfrac{1}{2}$$

4: (b) Let $U, V \subset J$ open.

Show: $\exists n \in \mathbb{Z}_{>0}$ s.t. $g^n(U) \cap V \neq \emptyset$

Set $\tilde{U} = h^{-1}(U)$ and $\tilde{V} = h^{-1}(V)$

where $h: I \to J$ is the homeomorph. that

conjugates f and g, i.e. $h \circ f = g \circ h$

$\tilde{U}, \tilde{V}$ are open as h is contin.

f is top. transit. $\Rightarrow \exists n \in \mathbb{Z}_{>0}$ s.t. $f^n(\tilde{U}) \cap \tilde{V} \neq \emptyset$,

i.e. $\exists x \in \tilde{U}$ with $f^n(x) \in \tilde{V}$

Set $y := h(x)$.

$\Rightarrow y \in h(\tilde{U}) = U$ and

$g^n(y) = g^n(h(x)) = h(f^n(x)) \in h(\tilde{V}) = V$